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On observer based stabilization of networked systems

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***Abstract*— Stabilizability of linear time invariant networked systems of general structure is studied with an observer-based approach. In the assumption of piecewise constant controls an average consensus network distributes input information to all agents enabling them to build local observers on the basis of which stabilizing gains are computed with recourse to standard centralized methodology. Stabilizability conditions are found for sampled-data and for discrete-time systems.**

I. INTRODUCTION

As established by the pioneeering work of [15], stabilizability of decentralized systems requires absence of so called unstable decentralized fixed modes. Under this condition - which plays at the local level the same role as stabilizability and detectability play at the centralized level - much of the effort has been directed to the design of compensators (static, dynamic, time varying, linear or non-linear) best suited to exploit the particular structure of the problem at hand [1],[6],[3],[4]. The presence of fixed modes can be algebraically characterized [2] through a combinatorial check which, although manageable [11], is more complex than stabilizability or detectability in the centralized framework. The real difficulty however is the actual computation of the compensator parameters. Even in the linear feedback case the block-diagonal structure of the gain matrix K introduces constraints that destroy the X, Y convex parametrization ($XK = Y$) holding in the centralized case. More recently, developments in networked control [12] and hybrid systems [8] have shown the convenience of observer based control schemes. In principle, if the state can be sufficiently well approximated by each agent, a centralized stabilizing gain (in the linear case) can be assumed to be commonly known to all agents. Exploiting this common knowledge, cooperating agents can solve the problem with considerably less computational effort. However when the control input is the sum of individual contributions from different agents, the centralized approach to observer design fails due to the lack of knowledge of control inputs used by other agents. Among the several works devoted to this problem, e.g. [7], [16] and refs. therein, two main approaches are available. The first is to reconstruct the input and the initial state from individual observations by left invertibility (or similar geometric properties) of the individual control systems [13],[14]. This however imposes restrictive conditions on the system matrices and even when these are satisfied, the reconstruction of the current state from agent i must enable her/him to stabilize the system, i.e. local stabilizability is needed. A second approach is to use communication among agents to supply the missing information. The approach offers potential advantages both in terms of robustness and computational effort. For example the stabilization of a formation of vehicles exchanging information with first neighbours has been recently formulated as a robust control problem [5]. However for systems of general structure a similar characterization is yet not known. The question of interest is to assess what can be achieved under a given - typically parsimonious - communication protocol for a system of general structure. In this paper a linear time invariant system of general structure is driven by the the input of r agents who privately observe the state, have no information of other agent's inputs, but exchange information through a consensus network, as described in Sec. III. In the assumption that inputs are kept piecewise constant, they can be propagated through the network and allow each agent to build a local state observer, on the basis of which the control is generated via the solution to a centralized stabilizaton problem. We consider the problem first in a sampled-data setting (Sec. IV-A) and then in the discrete-time setting (Sec. IV-B). In the first case a minimum information exchange rate to stabilize the system is determined. In the second case a necessary and sufficient condition for a stabilizing solution is found. It is shown that stabilizability under information exchange is possible only if a condition - generalizing absence of unstable decentralized fixed mode in the no-information case - is satsfied. A numerical example illustrates the results in Sec. V and conclusions are drawn in Sec. VI.

NOTATION. $\mathbf{1}$ is the vector with all components equal to 1, I_n is the identity matrix of order n, vec(Z) is the vector obtained by stacking the columns of matrix Z; Kronecker product is denoted by $\otimes$; the block-diagonal matrix with n identical blocks Z is denoted $diag_n(Z) = I_n \otimes Z$. When blocks are different we use $diag(Z_1 \dots Z_r)$ or, $diag(Z_i)$ if

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no ambiguity arises; $int(z)$ is the integer part of the real number z.

Acronym LMI stands for linear matrix inequality; DFM for decentralized fixed mode.

II. Definitions

The class of systems we consider in this paper is defined by

$$\begin{aligned} \partial x &= Ax + Bu, &(1)\\ y &= Cx, \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m, y \in \mathbb{R}^p &(2) \end{aligned}$$

where

$$B = [B_1 \,|\, \dots \,|\, B_r] \quad C = \begin{bmatrix} C_1 \\ \vdots \\ C_r \end{bmatrix} \tag{3}$$

with $B_i \in \mathbb{R}^{n\times m_i}$, $C_i \in \mathbb{R}^{p_i \times n}$ and $\sum p_i = p, \sum m_i = m$ and ∂ time-derivative in continuous time or delay operator in discrete time. We assume controls and observations are distributed among r agents. Each agent observes $y_i \in \mathbb{R}^{p_i}$, exchanges information with other agents and exerts control $u_i \in \mathbb{R}^{m_i}$. Agent i has no direct knowledge of u_j or y_j.

Assume agents communicate through a strongly connected graph $G = (V, E)$ whose nodes $V = \{1, 2, \dots, r\}$ represent agents and edges $E \subset V \times V$ represent transmission links. We denote $\mathcal{N}_i = \{j : \{i, j\} \in \mathcal{E}\}$ the neighbour nodes to i.

Assumption 1

i. (A, B) is stabilizable.
ii. (A, C_i) is detectable for all i.

Stabilizability is a necessary condition for distributed stabilization. As to local detectability, we assume the information exchange among agents concerns inputs not outputs. By keeping inputs constant over certain time intervals, this permits to exploit the averaging properties of consensus networks to enable agents reconstruct the state.

Remark 1: We do not assume (A, B_i) is stabilizable for otherwise the distributed stabilization problem could be entrusted to agent i and have a trivial solution - no communication would be necessary. Nevertheless, even if (A, B_i) is stabilizable for some i, exchange of information and distributed control can be highly beneficial to performance and robustness.

III. Consensus network

Let Adj be the adjacency matrix of G and W a weight matrix of the same size as Adj with the property

$$W_{ij} = W_{ji} \geq 0, \quad W\mathbf{1} = \mathbf{1}, \quad W_{ij} = 0 \text{ if } \ Adj_{ij} = 0.$$

Suppose we write on node i of G an initial *row-vector* $z_i'(0) = \{z_{1i}\, z_{2i} \dots z_{ni}\} \in \mathbb{R}^n$. Consider the overwriting scheme

$$z_{ki}(t+1) = z_{ki}(t) + \sum_{j\in\mathcal{N}_i} W_{ij}(z_{kj}(t) - z_{ki}(t)) \tag{4}$$
$$k = 1, \dots, n; \quad i = 1, \dots, r.$$

Each node updates itself by adding a weighted sum of the local discrepancies, i.e. differences between neighboring node values and its own.

By using Kronecker products, the scalar results in [17] can be vectorized.

Theorem 2: The vector consensus dynamics

$$\begin{aligned} \underline{z}(t+1) &= \mathbf{W}\underline{z}(t), \quad \underline{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} \in \mathbb{R}^{rn} &(5)\\ \mathbf{W} &= W \otimes I_n \in \mathbb{R}^{rn\times rn} \end{aligned}$$

displays the averaging property

$$\begin{aligned} z_i(t) &= \mathbf{c}_i \mathbf{W}^t \underline{z}(0) = \frac{1}{r}\sum_{k=1}^{r} z_k(0) + \epsilon_i(t)\\ \epsilon_i(t) &= \mathbf{c}_i([W^t - J] \otimes I_n)\underline{z}(0)\\ \sum_{i=1}^{r} \epsilon_i(t) &= 0, \forall t \geq 0\\ ||\epsilon_i(t)|| &\leq \rho^t ||\underline{z}(0)|| \end{aligned}$$

where $\mathbf{c}_i = [0 \dots I_r \dots 0] \in \mathbb{R}^{r\times nr}$, I_r in position i, $J = \mathbf{1}\mathbf{1}'/r$ and $\rho < 1$ is the second largest eigenvalue of W in absolute value.

Proof: Let

$$\begin{aligned} Z(t) &= \begin{bmatrix} z_1(t) \,| & \dots & | \, z_r(t) \end{bmatrix}\\ &= \begin{bmatrix} z_{11}(t) & \dots & z_{1r}(t) \\ \vdots & & \vdots \\ z_{n1}(t) & \dots & z_{nr}(t) \end{bmatrix} \in \mathbb{R}^{n\times r}. \end{aligned}$$

Using $\sum_{j\in\mathcal{N}_i} W_{ij} = 1$, recursion (4) can be written

$$z_{ki}(t+1) = \sum_{j\in\mathcal{N}_i} W_{ij} z_{kj}(t)$$

or

$$Z'(t+1) = WZ'(t). \tag{6}$$

Rewriting as $Z(t+1) = I_n\, Z(t)W$ and using a known property of Kronecker product we get

$$vec(Z(t+1)) = W \otimes I_n\, vec(Z(t)).$$

Recognizing $vec(Z) = \underline{z}$ we get (5). Since $z_i = \mathbf{c}_i Z'$

$$\begin{aligned} z_i(t) &= \mathbf{c}_i W^t Z'(0) = \mathbf{c}_i J Z'(0) + \mathbf{c}_i(W^t - J)Z'(0)\\ &= \frac{1}{r}\sum_{i=1}^{r} z_i(0) + \mathbf{c}_i([W^t - J] \otimes I_n)\underline{z}(0). \end{aligned}$$

Moreover, $\mathbf{c}_i = c_i' \otimes I_n$ where c_i' is the i-th row of I_r. Therefore

$$\sum_i \mathbf{c}_i = \sum_i (c_i' \otimes I_n) = (\sum_i c_i') \otimes I_n = \mathbf{1}' \otimes I_n$$

and distributing $\otimes$

$$\sum_i \mathbf{c}_i([W^t - J] \otimes I_n)\underline{z}(0) = [\mathbf{1}'W^t - \mathbf{1}'J] \otimes I_n \underline{z}(0) = 0.$$

Finally

$$\begin{aligned}||\epsilon_i(t)|| &= ||\mathbf{c}_i([W^t - J]\otimes I_n)\underline{z}(0)|| = ||(W^t - J)\underline{z}(0)|| \\ &\leq ||W^t - J||\cdot||\underline{z}(0)||\end{aligned}$$

and the characterization $||W^t - J|| = \rho^t$ is well known. ■

Remark 3: Essentially (6) describes the matrix-form dynamics of r parallel consensus protocols. The equivalent vector-form (5) is derived for its convenience in the developments to follow.

Our approach is based on a consensus network to help each agent reconstruct the input used by other agents, in the assumption that over intervals of length μ all agents keep their inputs constant. Model (1-3) together with W will henceforth be called a distributed control system (S, W).

Problem Statement: Find conditions for which it is possible to stabilize (S, W) by piecewise constant control on the basis of information exchanged among r agents over a consensus network.

It turns out that the problem is more easily solved for sampled-data systems than for discrete-time systems.

IV. Distributed stabilizing controller

A. Sampled data systems

Let $\partial x = \dot{x}$ in (1,2) and consider the continuous time system

$$\begin{aligned}\dot{x} &= A_c x + B_c u \\ y &= Cx.\end{aligned}$$

with (A_c, B_c) stabilizable and (A_c, C_i) detectable for all i. The associated sampled-data system is

$$\begin{aligned}x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k\end{aligned}$$

where $A = e^{A_c\Delta}$, $B = \int_0^\Delta e^{A_c(\Delta-\tau)}B_c d\tau$. By the results in [9], stabilizability of (A_c, B_c) and detectability of (A_c, C_i) is preserved for (A, B) and (A, C_i) for almost all sampling periods. Suppose agents are connected through a consensus network like (5) with initial condition

$$\underline{z}(0) = r\begin{bmatrix} B_1 u_{1,k} \\ \vdots \\ B_r u_{r,k}\end{bmatrix}.$$

If updates in (5) occur at a "chat-rate" $\frac{\mu}{\Delta}$ - that is μ times per sample-time - from Thm 2 agent i receives an estimate of Bu_k

$$v_{i,h} = Bu_k + \epsilon_{i,h}, \quad ||\epsilon_{i,h}|| \leq \rho^\sigma ||rBu_k|| \tag{7}$$

at each time $t = h\frac{\mu}{\Delta}$, $h = 0, 1\ldots$ where $\sigma = h - int\left(\frac{h-1}{\mu}\right)\mu$. Hence at time $t = k\Delta$

$$v_{i,k} = Bu_k + \epsilon_{i,k}, \quad ||\epsilon_{i,k}|| \leq \rho^\mu ||rBu_k||. \tag{8}$$

Suppose the state x_k is not known. Consider a "node" observer

$$\lambda_{i,k+1} = A\lambda_{i,k} + v_{i,k} + R_i\left(C\lambda_{i,k} - y_{i,k}\right)$$

and define an estimation error

$$e_{i,k} = \lambda_{i,k} - x_k$$

Subtracting $x_{k+1} = Ax_k - Bu_k$

$$\begin{aligned}e_{i,k+1} &= (A + R_iC_i)e_{i,k} + v_{i,k} - Bu_k \\ &= (A + R_iC_i)e_{i,k} + \epsilon_{i,k}, \quad ||\epsilon_{i,k}|| \leq \rho^\mu ||rBu_k||\end{aligned}$$

Suppose

$$u_{i,k} = K_i\lambda_{i,k} = K_i(e_{i,k} + x_k)$$

then, letting

$$K = \begin{bmatrix} K_1 \\ \vdots \\ K_r\end{bmatrix} \qquad \mathbf{K} = \begin{bmatrix} K_1 & & \\ & \ddots & \\ & & K_r\end{bmatrix},$$

$$\begin{aligned}x_{k+1} &= (A + BK)x + B\mathbf{K}\mathbf{e}_k \\ \mathbf{e}_{k+1} &= \mathbf{A}\mathbf{e}_k + \widehat{\epsilon}_k,\end{aligned}$$

where $\widehat{\epsilon_k} = vec(\epsilon_{i,k})$, $\mathbf{A} = diag(A + R_iC_i)$.

Letting $\xi = \begin{bmatrix} x \\ \mathbf{e}\end{bmatrix}$

$$\xi_{k+1} = \begin{bmatrix} A + BK & B\mathbf{K} \\ 0 & \mathbf{A}\end{bmatrix}\xi_k + \begin{bmatrix} 0 \\ \widehat{\epsilon_k}\end{bmatrix}. \tag{9}$$

Since (A, B) is stabilizable, and (A, C_i) is detectable, $i = 1, \ldots, r$ there exist gains R_i and K such that the matrix in (9) is Schur-stable. Let

$$\widetilde{A} = \begin{bmatrix} A + BK & B\mathbf{K} \\ 0 & \mathbf{A}\end{bmatrix} \tag{10}$$

Theorem 4: For any stabilizing gains R_i and K, there exists μ such that all the trajectories of system (9) asymptotically converge to the origin.

Proof: Since $\widetilde{A}$ is Schur-stable, then $\widetilde{A}\Omega \subset \lambda\Omega$ for some ellipsoid Ω and for some $\lambda \in (0, 1)$. Therefore $\forall\xi \in \Omega$, and $\forall\widehat{\lambda} \in (\lambda, 1)$ there exists a sufficiently small ε such that $\widetilde{A}\xi + \varepsilon\,\|\xi\|\,\mathcal{B} \subset \widetilde{A}\Omega + \varepsilon\beta\ \mathcal{B} \subset \widehat{\lambda}\Omega$, with $\mathcal{B}$ a ball in the appropriate space and where $\beta = \max_{\xi\in\Omega}\|\xi\|$. Since $\epsilon_{i,k} \in \rho^\mu\,\|rBK\|\,\beta\mathcal{B}$ then $\widehat{\epsilon_k} \in \rho^\mu\,\|rBK\|\,r^{\frac{1}{2}}\beta\mathcal{B}$, by definition of $\widehat{\epsilon_k}$, and the result follows. ■

For given R_i, K it is of interest to estimate the minimum chat-rate guaranteeing Schur-stability.

Theorem 5: Let R_i, K_i be given. For any positive scalars $p, \alpha, \lambda < 1$ satisfying the LMI

$$\begin{aligned}&Q > 0 \\ &(1 + p)\tilde{A}Q\tilde{A}' + (1 + p^{-1})\alpha^2\begin{bmatrix} 0 & 0 \\ 0 & 1\end{bmatrix} < \lambda^2 Q\end{aligned}$$

the chat-rate of $\mu = \frac{ln\,\gamma}{ln\,\rho}$, $\gamma = \alpha/\,\|rBK\|\,r^{\frac{1}{2}}$ times per sample guarantees Schur-stabilty of (S, W).

Proof: Let $P, Q > 0$ and let $\mathcal{E}(Q) = \{x : x'Q^{-1}x \leq 1\}$. We recall [10]

$$\begin{array}{l} \mathcal{E}(P) \subset \mathcal{E}(Q) \Leftrightarrow P < Q \\ \tilde{A}\mathcal{E}(Q) = \mathcal{E}(\tilde{A}Q\tilde{A}') \\ \mathcal{E}(Q) + \mathcal{E}(P) \subset \mathcal{E}((1+p)Q + (1+p^{-1})P), \ \forall p > 0. \end{array}$$

Schur-stability of (9) holds if for some Ω

$$\tilde{A}\Omega + \alpha \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathcal{B} \subset \lambda\Omega.$$

Letting $\Omega = \mathcal{E}(Q)$, $\mathcal{B} = \mathcal{E}(I)$ in the above inclusion we get the LMI in the statement. Recalling that $\widehat{\epsilon_k} \in \rho^\mu \|rBK\| r^{\frac{1}{2}} \beta \mathcal{B}$ the conclusion follows. ■

B. Discrete-time systems

Let $\partial x = x(t+1)$ in (1,2) and consider the discrete-time system

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t). \end{aligned}$$

Let μ be a positive integer and introduce intervals

$$T_k = \{t : \mu k \leq t < \mu k + \mu)\}, \quad k = 0, 1, 2 \ldots$$

Let $t_k = \mu k$ and $u_k = u(t_k)$ be a control vector in $\mathbb{R}^m$ that remains constant over the interval T_k. The μ-lifted state evolves as

$$\begin{aligned} x_{k+1} &= A^\mu x_k + [I \,|\, A \,|\, \ldots \,|\, A^{\mu-1}] \begin{bmatrix} Bu_k \\ \vdots \\ Bu_k \end{bmatrix} \\ &= A^\mu x_k + A_\mu B u_k \end{aligned} \qquad (11)$$

having set $A_\mu = \sum_{\tau=1}^{\mu} A^{\tau-1}$. Let $u_{i,k} \in \mathbb{R}^{m_i}$ be the ordered components of the i-th subvector of $u_k \in \mathbb{R}^m$ ($m_1 + \cdots + m_r = m$) that agent i keeps constant over T_k. Suppose at each $t \in T_k$ agent i receives an estimate $v_i(t)$ of Bu_k from a consensus network like (5). If each node of the network is initialized at $z_i(\mu k) = rB_i u_{i,k} \in \mathbb{R}^n, i = 1 \ldots r$, from Thm 2 we get

$$\begin{aligned} v_i(\mu k + \tau - 1) &= \mathbf{c}_i \mathbf{W}^{\tau-1} \mathbf{B} u_k, \, \tau = 1, \ldots, \mu \quad (12) \\ \mathbf{B} &= diag(rB_i). \end{aligned}$$

If the state $x(\mu k) = x_k$ is not known in (11) it can be estimated by a "node" observer

$$\lambda_{i,k+1} = A^\mu \lambda_{i,k} + A_\mu v_{i,k} + R_i(C_i\lambda_{i,k} - y_{i,k}) \qquad (13)$$

with input $v_{i,k}$ set equal to the last estimate of Bu_k available to agent i after communication with its neighbours over T_k. Using (12) in (13) we get

$$\begin{aligned} \lambda_{i,k+1} &= A^\mu \lambda_{i,k} + A_\mu \mathbf{c}_i \mathbf{W}^{\mu-1} \mathbf{B} u_k \\ &+ R_i(C_i \lambda_{i,k} - y_{i,k}). \end{aligned} \qquad (14)$$

Defining an estimation error

$$e_{i,k} = \lambda_{i,k} - x_k$$

with x_k evolving as in (11), we get

$$\begin{aligned} e_{i,k+1} &= (A^\mu + R_i C_i) e_{i,k} \\ &+ A_\mu (\mathbf{c}_i \mathbf{W}^{\mu-1} \mathbf{B} - B) u_k. \end{aligned} \qquad (15)$$

Combining (11,15)

$$\begin{aligned} \begin{bmatrix} x_{k+1} \\ e_{i,k+1} \end{bmatrix} &= \begin{bmatrix} A^\mu & 0 \\ 0 & A^\mu + R_i C_i \end{bmatrix} \begin{bmatrix} x_k \\ e_{i,k} \end{bmatrix} \\ &+ \begin{bmatrix} A_\mu B \\ A_\mu (\mathbf{c}_i \mathbf{W}^{\mu-1} \mathbf{B} - B) \end{bmatrix} u_k. \end{aligned}$$

Introducing $B_\mu = A_\mu B$ and

$$\begin{aligned} \mathbf{A}^\mu &= diag(A^\mu + R_i C_i) \\ \mathbf{B}_\mu &= \begin{bmatrix} A_\mu (\mathbf{c}_1 \mathbf{W}^{\mu-1} \mathbf{B} - B) \\ \vdots \\ A_\mu (\mathbf{c}_r \mathbf{W}^{\mu-1} \mathbf{B} - B) \end{bmatrix} \end{aligned}$$

the aggregated system evolves as

$$\begin{bmatrix} x_{k+1} \\ \mathbf{e}_{k+1} \end{bmatrix} = \begin{bmatrix} A^\mu & 0 \\ 0 & \mathbf{A}^\mu \end{bmatrix} \begin{bmatrix} x_k \\ \mathbf{e}_k \end{bmatrix} + \begin{bmatrix} B_\mu \\ \mathbf{B}_\mu \end{bmatrix} u_k. \qquad (16)$$

Suppose there is a procedure (to be specified later) to select $K = [K_1 \ldots K_r]$, $K_i \in \mathbb{R}^{m_i \times n}$ which is common knowledge to all agents. Then agent i can use a feedback policy

$$u_{i,k} = K_i \lambda_{i,k} = K_i (x_k + e_{i,k}) \qquad (17)$$

which results in an aggregated input

$$u_k = K x_k + \mathbf{K} \, \mathbf{e}_k.$$

Substituting in (16)

$$\begin{aligned} \begin{bmatrix} x \\ \mathbf{e} \end{bmatrix}^+ &= \begin{bmatrix} A^\mu & 0 \\ 0 & \mathbf{A}^\mu \end{bmatrix} \begin{bmatrix} x \\ \mathbf{e} \end{bmatrix} \\ &+ \begin{bmatrix} B_\mu \\ \mathbf{B}_\mu \end{bmatrix} [K \,|\, \mathbf{K}] \begin{bmatrix} x \\ \mathbf{e} \end{bmatrix} \end{aligned} \qquad (18)$$

we arrive at the conclusion

Theorem 6: The distributed stabilization problem with piecewise constant control and observations every μ steps has a linear feedback solution if and only if there exist R, K stabilizing

$$M = \begin{bmatrix} A^\mu & 0 \\ 0 & \mathbf{A}^\mu \end{bmatrix} + \begin{bmatrix} B_\mu \\ \mathbf{B}_\mu \end{bmatrix} [K \,|\, \mathbf{K}]. \qquad (19)$$

Notice that this conclusion not necessarily implies the assumption of local detectability or local stabilizability.

It is possible to generalize the decentralized stabilizability condition given in [15] by the following

Theorem 7: The distributed stabilization problem with piecewise constant control has a linear feedback solution only if the triple (F_μ, G_μ, T) with

$$
\begin{aligned}
F_\mu &= \begin{bmatrix} A^\mu & 0 \\ 0 & \mathbf{A}^\mu \end{bmatrix} \in \mathbb{R}^{(r+1)n\times(r+1)n} \\
G_\mu &= \begin{bmatrix} B_\mu \\ \mathbf{B}_\mu \end{bmatrix} \in \mathbb{R}^{(r+1)n\times m} \\
T &= \begin{bmatrix} I & I & 0 & \dots & 0 \\ I & 0 & I & \dots & 0 \\ \vdots & & & \ddots & \\ I & & \dots & & I \end{bmatrix} \in \mathbb{R}^{m\times(r+1)n}
\end{aligned}
$$

has no unstable DFM wrt to the partition

$$
\begin{aligned}
G_\mu &= [G_1 \,|\, \dots \,|\, G_r], \quad G_i \in \mathbb{R}^{(r+1)n\times m_i} \\
T &= \begin{bmatrix} T_1 \\ \vdots \\ T_r \end{bmatrix}, \quad T_i \in \mathbb{R}^{m_i\times(r+1)n}, i = 1, \dots, r.
\end{aligned}
$$

Proof: Consider the identity

$$
\begin{aligned}
& \begin{bmatrix} B_\mu \\ \mathbf{B}_\mu \end{bmatrix} [K \,|\, \mathbf{K}\,] \\
= & \begin{bmatrix} B_\mu \\ \mathbf{B}_\mu \end{bmatrix} \begin{bmatrix} K_1 & K_1 & 0 & \dots & 0 \\ K_2 & 0 & K_2 & \dots & 0 \\ \vdots & & & \ddots & \\ K_r & & \dots & & K_r \end{bmatrix} \\
= & \begin{bmatrix} B_\mu \\ \mathbf{B}_\mu \end{bmatrix} diag(K_1 \dots K_r)T
\end{aligned}
$$

The conclusion follows from Theorem 1 of [15]. ■

Notice that for $\mu = 1$ $(S, W) = (S, I)$, i.e. there is no exchange of information over the communication network and our distributed system collapses to the decentralized case studied in [15]. In that case, stabilizability is equivalent to absence of unstable DFM of (S, I). Thus our results generalize [15].

Notice that Thm 6 refers to the case in which eq (17) is used in (16), i.e. the case of *static* linear feedback. If one allowed for *dynamic* linear feedback, then the no unstable DFM condition of Thm 7 would become necessary *and sufficient* for distributed stabilizability of (S, W) with piecewise constant inputs.

V. Example

The example below considers distributed stabilization of a 5-agent process, with agent i described by (A, B_i, C_i). Since output information is not exchanged, we take for simplicity $C_i = C, \forall i$. Specifically, we consider a $r = 5$ agent system of order $n = 10$ described (in aggregated form) by

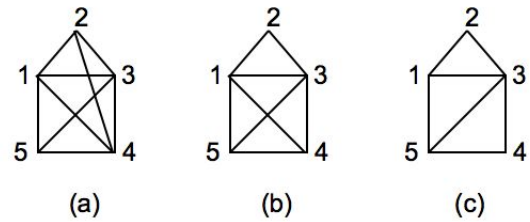

Fig. 1. Communication graphs

$$
A = \frac{1}{5}\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}
$$

$$
B = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & 1 & 1 & 0 & 1 & 0 & -1 & 1 \\
1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 1 \\
-1 & 1 & 0 & 0 & -1 & -1 & 1 & 0 & -1 & 1 \\
1 & -1 & -1 & 0 & 0 & 1 & 0 & -1 & 1 & 1 \\
0 & 1 & -1 & 0 & -1 & -1 & 1 & 1 & 1 & -1 \\
-1 & 0 & 1 & -1 & 1 & -1 & 0 & 1 & 0 & 1 \\
-1 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 \\
-1 & 1 & 1 & -1 & 0 & 0 & -1 & 1 & 1 & 0 \\
-1 & 1 & 1 & -1 & -1 & 1 & 1 & 0 & -1 & -1
\end{bmatrix}
$$

$$
C_i = \begin{bmatrix}
3 & 4 & 1 & 0 & -2 & -2 & -3 & 3 & 3 & -3 \\
5 & 1 & 4 & 0 & -1 & 5 & 0 & -2 & -3 & 3 \\
2 & 4 & 0 & -2 & -4 & -4 & -1 & -4 & 2 & 0 \\
2 & 1 & 3 & 1 & 2 & 0 & 1 & 1 & 4 & 1 \\
-3 & -3 & -2 & 1 & 3 & 2 & 3 & -2 & 0 & -2 \\
-1 & 1 & -2 & 4 & -3 & -2 & 0 & -1 & -4 & 0 \\
4 & -1 & 0 & -1 & 2 & 2 & 2 & -1 & -4 & 5 \\
5 & 3 & 0 & 3 & 3 & 2 & 2 & 1 & 4 & -1 \\
-1 & 2 & -1 & -3 & 0 & 3 & 1 & -4 & 0 & 1
\end{bmatrix}
$$

The first 2 columns of B (=B_1) are assigned to agent 1; the next 2 columns of B (=B_2) to agent 2 etc; (A, B_i) is *not* stabilizable for $i = 1, \dots, 5$; (A, B) is reachable, (A, C_i) is observable, with C_i common to all agents. A is unstable with $\max_i |\lambda_i(A)| = 1.1617$. We consider three communication graphs of decreasing connectivity, Fig 1 (a-c).

For each graph the maximum eigenvalue of the closed-loop matrix M (eq (19)) in abs-value is shown in Fig 2, for increasing values of μ. Matrices R_i, K have been determined as $R_i = Y_i X_i^{-1}$ and $K = Y\hat{X}^{-1}$ where

$$
\begin{aligned}
& X_i = arg \min_{Y_i, X>0} ||X|| \\
& \begin{bmatrix} X & A'^\mu X + C_i' Y_i \\ (A'^\mu X + C_i' Y_i)' & X \end{bmatrix} > 0
\end{aligned}
$$

and

$$
\begin{aligned}
& \hat{X} = arg \min_{Y, X>0} ||X|| \\
& \begin{bmatrix} X & A^\mu X_i + B_\mu Y \\ (A^\mu X + B_\mu Y)' & X \end{bmatrix} > 0.
\end{aligned}
$$

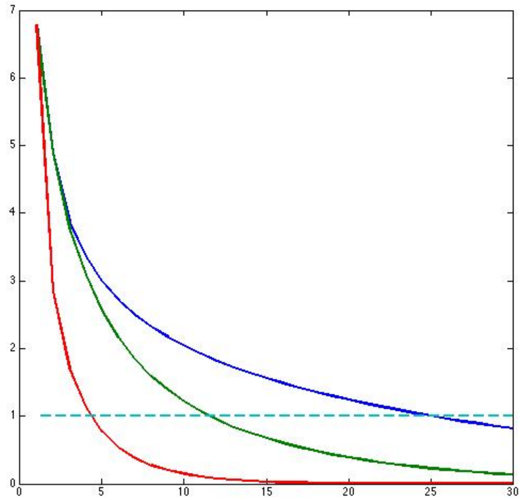

Fig. 2. $\max_i |\lambda_i|$ vs μ. Left: (a); Center: (b); Right: (c)

Notice that stabilization is obtained by "freezing" inputs for about 5 periods in case (a) 12 periods in case (b) and 25 periods in case (c).

VI. CONCLUSION

Exchange of information over a consensus network permits to design stabilizing controllers for distributed control systems of general structure. In the assumption that inputs are kept piecewise constant, they can be propagated through the network and allow each agent to build a local state observer. On the basis of the local estimates, a linear feedback controller for each agent resulting in stability of the overall system can be designed. In the case of sample-data systems the feedback gains always exist under centralized stabilizability and local detectability provided the information exchange is fast enough. It is noteworthy that such gains can be computed by solving a centralized problem, via standard off-line LMI parametrization; whereas synthesising a controller without information exchange might require a much more complex (e.g., non convex and possibly on-line) computation. Here is an instance of the trade-off between amount of local elaboration vs. amount of transmitted information - distributed average consensus being amongst the most parsimonious transmission mechanisms available. In the case of discrete-time systems, a stabilizing controller requires satisfaction of a condition generalizing absence of unstable DFM to the case of exchange of information. Unlike the sampled-data case, local feedback gains *may* be computed from a centralized stabilizing controller (as shown in the numerical example) or may not. In that case one has to resort to a controller of the same structure as in the no-information case, e.g. a dynamic compensator with a block diagonal gain matrix. When local stabilizability or local detectability fail, the absence of unstable DFM in the no-information case is a necessary (and sufficient) condition for existence of a stabilizing controller.

Whether the same condition is necessary in presence of information exchange remains to be explored. Our conjecture is that it is not.

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