Rare-event probability estimation with application to air traffic

by

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Rare-event probability estimation with application to air traffic

- Motivation
- Advanced air traffic example
- Interacting Particle System (IPS)
- Hierarchical Hybrid IPS
- Results for air traffic example
- Conclusions
The capacity ‘wall’

Source: PRC 1 report

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The capacity “wall” is a safety “wall”

- Capacity relates directly to safety
- The question usually is: how to increase capacity whilst at the same time manage the safety?
- The answer is: by improving both capacity and safety per flight
Safety feedback based design

Air traffic operation design

Safety / Capacity Assessment
iFly

- Innovative project for European Commission
  - Follow-up of HYBRIDGE project
  - Partners: 11 universities + 7 from AirTraffic/Aviation
  - NLR is coordinator
  - iFly project duration: May 2007- August 2011
  - Web site  http://iFly.nlr.nl
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Autonomous Mediterranean Free Flight (AMFF)

- Future concept developed for traffic over Mediterranean area
- Aircrew gets freedom to select path and speed
- In return aircrew is responsible for self-separation
- Each a/c equipped with an Airborne Separation Assistance System
- In AMFF, conflicts are solved one by one (pilot preference)
- Can AMFF safely accommodate high traffic demand?
Development of MC simulation model

- Hazard identification
- Defining the relevant Agents
- Developing Petri net for each Agent
- Connecting Agent Petri nets
- Parametrization, Verification & Calibration
- Verification & Calibration
Agents in SHS model of AMFF
### Dimensional analysis of SHS model of AMFF

<table>
<thead>
<tr>
<th>Agent</th>
<th># of product places</th>
<th>Maximum colour product state space</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aircraft</td>
<td>$24^N$</td>
<td>$IR^{13N}$</td>
</tr>
<tr>
<td>Pilot-Flying (PF)</td>
<td>$490^N$</td>
<td>$IR^{28N}$</td>
</tr>
<tr>
<td>Pilot-not-Flying (PNF)</td>
<td>$7^N$</td>
<td>$IR^{3N}$</td>
</tr>
<tr>
<td>AGNC</td>
<td>$(15 \times 2^{16})^N$</td>
<td>$IR^{45N}$</td>
</tr>
<tr>
<td>ASAS</td>
<td>$48^N$</td>
<td>$IR^{37N+21N^2}$</td>
</tr>
<tr>
<td>Global CNS</td>
<td>16</td>
<td>${}$</td>
</tr>
<tr>
<td><strong>PRODUCT</strong></td>
<td>$\approx 16 \times (3.88 \times 10^{12})^N$</td>
<td>$IR^{126N+21N^2}$</td>
</tr>
</tbody>
</table>
Eight aircraft encounter
Approaches in Reach Probability Computation

- Markov Chain (MC) approximation (Prandini&Hu, 2006)
- Dynamic Programming (DP) approach (Abate, Amin, Prandini, Lygeros & Sastry, 2006)
- Interacting Particle System (IPS) approach (Cerou et al., 2005)
- Hybrid IPS (Krystul & Blom, 2005, 2006)

Large scale SHS may cause scalability problems
- State space is too large to handle
- Relevant mode switching is rare
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SHS Reach probability

We consider a time-homogeneous strong Markov process which is a generalised stochastic hybrid process \( \{x_t, \theta_t\} \), with \( \{x_t\} \) assuming values in \( \mathbb{R}^n \) and \( \{\theta_t\} \) assuming values in discrete set \( \mathbb{M} \). The first component of \( \{x_t\} \) equals \( t \) and the other components of \( \{x_t\} \) form an \( \mathbb{R}^{n-1} \) valued cadlag process \( \{s_t\} \). The problem considered is to estimate the probability that \( \{s_t\} \) hits a given “small” closed subset \( D \subset \mathbb{R}^{n-1} \) within a given time period \( [0,T) \), i.e. \( P(\tau < T) \) with \( \tau = \inf\{t > 0; s_t \in D\} \).
Reach Probability Factorization

Assume nested sequence of closed subsets

\[ D = D_m \subset D_{m-1} \subset \ldots \subset D_1, \]

with the constraints that \( P(s_0 \in D_1) = 0 \) and each component of \( \{s_t\} \) that may hit \( D_k, k = 1, 2, \ldots m \) has continuous paths \( P \)-a.s.

We set \( \tau_0 = 0 \) and define \( \tau_k, k = 1, \ldots, m \), as the first moment that \( \{s_t\} \) hits subset \( k \), i.e. \( \tau_k = \inf\{t > 0; s_t \in D_k\} \)

Then (e.g. L’Ecuyer et al., 2006):

\[
P(\tau < T) = \prod_{k=1}^{m} \gamma_k \quad \text{with} \quad \gamma_k \triangleq P(\tau_k < T \mid \tau_{k-1} < T)
\]
Interacting Particle System (IPS)

- Define a sequence of conflict levels decreasing in urgency \((D_k\)'s\)
  - Most urgent level represents collision \((D_m = D)\)

- Simulate \(N_p\) particles; initially all outside \(D_1\) (less urgent level)

- Freeze each particle that reaches the next urgent level before \(T\)

- Make \(N_p\) copies of frozen particles

- Repeat this until the most urgent level has been reached

- Count the simulated fraction \(\tilde{\gamma}_k\) that reaches level \(k\)

- Estimated collision risk = \(\tilde{\gamma}_1 \times \tilde{\gamma}_2 \times \tilde{\gamma}_3 \times \ldots \times \tilde{\gamma}_m\)
IPS convergence

Cerou, Del Moral, Legland and Lezaud (2002, 2005) have shown that the product of these fractions \( \tilde{\gamma}_k \) forms an unbiased estimate of the probability of \( \{s_t\} \) to hit the set \( D \) within the time period \( [0,T) \), i.e.

\[
\mathbb{E}[\prod_{k=1}^{m} \tilde{\gamma}_k] = \prod_{k=1}^{m} \gamma_k = P(\tau < T)
\]

In addition there is a bound on the \( L^1 \) estimation error, i.e.:

\[
\mathbb{E}(\prod_{k=1}^{m} \tilde{\gamma}_k - \prod_{k=1}^{m} \gamma_k) \leq \frac{c_p}{\sqrt{N_p}}
\]
Hybrid IPS versions

1. Importance switching (Krystul&Blom, 2005)

2. Rao-Blackwellization, using exact equations for \( \{ \theta_t \} \) and particles for Euclidian state (Krystul&Blom, 2006)

- Both handle rare mode switching well
- New large scale SHS scalability problem
  - Combinatorially many discrete modes
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Hierarchical Hybrid IPS (HHIPS)

✓ Define an aggregated mode process \{ \kappa_t \}

\[ \kappa_t = \kappa \text{ if } \theta_t \in M_k \]

with \{M_k, \kappa \in K\} a partition of \( M \)

✓ Apply Importance switching to \{ \kappa_t \}

✓ Rao-Blackwellization, i.e. use exact equations for \{ \kappa_t \}
and particles for the other process elements \{x_t, \theta_t\}
Hierarchical Hybrid IPS (HHIPS)

Step 0 generates per aggregation mode value $\kappa \in \mathcal{K}$, $N_p$ initial particles for $k=0$, and then starts the cycling through steps 1 through 3:

Step 1 extrapolates each $(x_t, \theta_t, \kappa_t)$-particle from $\tau_{k-1} \land T$ to $\tau_k \land T$

Step 2 evaluates the $(x_{\tau_k}, \theta_{\tau_k}, \kappa_{\tau_k})$ particles that have arrived at $Q_k$

Step 3 resamples from the $(x_{\tau_k}, \theta_{\tau_k}, \kappa_{\tau_k})$ particles that have arrived at $Q_k$

set $k := k + 1$ and go to step 1
Step 1 extrapolates each particle from $\tau_{k-1} \wedge T$ to $\tau_k \wedge T$ in time step of length $h$, using importance switching for the new $\kappa$ - value and $\kappa$ - conditional sampling of a new $\theta$ – value. For the latter use is made of the following theorem:

Theorem 1 ( $\kappa$ - conditional $\theta$ –prediction )

Let $\tau$ be an arbitrary stopping time, then

$$p_{\theta_{\tau+h}|x,\theta,\kappa_{\tau+h}}(\eta | x, \theta, \kappa) = \frac{1_{\mathbb{M}_\kappa}(\eta) p_{\theta_{\tau+h}|x,\theta,\kappa}(\eta | x, \theta)}{\sum_{\eta' \in \mathbb{M}} 1_{\mathbb{M}_\kappa}(\eta') p_{\theta_{\tau+h}|x,\theta,\kappa}(\eta' | x, \theta)}$$
Step 3  resamples from the particles that have arrived at $Q_k$
In order to draw $N_p$ samples per $\kappa$ – value, use is made of the following hierarchical interaction theorem:

**Theorem 2 (Hierarchical interaction)**

If $p_{\kappa_{\tau+h}}(\kappa) > 0$ for arbitrary stopping time $\tau$, then

$$p_{x_\tau, \theta_\tau | \kappa_{\tau+h}}(dx, \theta | \kappa) = \sum_{\eta \in \mathcal{M}_\kappa} p_{\theta_{\tau+h} | x_\tau, \theta_\tau}(\eta | x, \theta) \cdot p_{x_\tau, \theta_\tau}(dx, \theta) / p_{\kappa_{\tau+h}}(\kappa)$$

$$p_{\kappa_{\tau+h}}(\kappa) = \sum_{\theta \in \mathcal{M}} \int_{\mathbb{R}^n} \sum_{\eta \in \mathcal{M}_\kappa} p_{\theta_{\tau+h} | x_\tau, \theta_\tau}(\eta | x, \theta) \cdot p_{x_\tau, \theta_\tau}(dx, \theta)$$
Key extensions of HHIPS over IPS for SHS

- Embedding of an aggregation mode process;
- Particles are maintained per aggregation mode;
- Importance switching of aggregation mode is used for the conditional prediction of SHS particles;
- Hierarchical interaction is used for the resampling of particles that reached \( Q_k \triangleq (0, T) \times D_k \times \mathbb{M}, \ k = 1,.., m - 1 \).
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Scenarios

- Two aircraft encounter using HHIPS
- Eight aircraft encounter using IPS
- Random traffic high density using IPS
Air traffic safety related events

<table>
<thead>
<tr>
<th>Event</th>
<th>MTC</th>
<th>STC</th>
<th>MSI</th>
<th>NMAC</th>
<th>MAC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prediction time (minutes)</td>
<td>8</td>
<td>2.5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Horizontal distance (Nm)</td>
<td>4.5</td>
<td>4.5</td>
<td>4.5</td>
<td>1.25</td>
<td>0.054</td>
</tr>
<tr>
<td>Vertical distance (ft)</td>
<td>900</td>
<td>900</td>
<td>900</td>
<td>500</td>
<td>131</td>
</tr>
</tbody>
</table>

MTC  =  Medium Term Conflict  
STC  =  Short Term Conflict  
MSI  =  Minimum Separation Infringement  
NMAC =  Near Mid-Air Collision  
MAC  =  Mid-Air Collision
Sequence of conflict levels for air traffic

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D_k$ (Nm)</td>
<td>4.5</td>
<td>4.5</td>
<td>4.5</td>
<td>4.5</td>
<td>2.5</td>
<td>1.25</td>
<td>0.5</td>
<td>0.054</td>
</tr>
<tr>
<td>$h_k$ (ft)</td>
<td>900</td>
<td>900</td>
<td>900</td>
<td>900</td>
<td>900</td>
<td>500</td>
<td>250</td>
<td>131</td>
</tr>
<tr>
<td>$\Delta_k$ (min)</td>
<td>8</td>
<td>2.5</td>
<td>1.5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

- **Medium Term Conflict**
- **Short Term Conflict**
- **Minimum Separation Conflict**
- **Near Mid-Air Collision (NMAC)**
- **Mid-Air Collision**
\( \tilde{\gamma}_k \) values estimated by HHIPS for Two-aircraft scenario

<table>
<thead>
<tr>
<th>( k )</th>
<th>Run 1</th>
<th>Run 2</th>
<th>Run 3</th>
<th>Run 4</th>
<th>Run 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>0.991</td>
<td>1.000</td>
</tr>
<tr>
<td>2</td>
<td>5.77E-04</td>
<td>5.64E-06</td>
<td>6.24E-06</td>
<td>5.04E-06</td>
<td>6.13E-06</td>
</tr>
<tr>
<td>3</td>
<td>6.40E-03</td>
<td>7.25E-01</td>
<td>7.20E-01</td>
<td>6.84E-01</td>
<td>7.66E-01</td>
</tr>
<tr>
<td>4</td>
<td>0.566</td>
<td>0.569</td>
<td>0.596</td>
<td>0.540</td>
<td>0.608</td>
</tr>
<tr>
<td>5</td>
<td>0.344</td>
<td>0.256</td>
<td>0.223</td>
<td>0.401</td>
<td>0.198</td>
</tr>
<tr>
<td>6</td>
<td>0.420</td>
<td>0.452</td>
<td>0.402</td>
<td>0.459</td>
<td>0.429</td>
</tr>
<tr>
<td>7</td>
<td>0.801</td>
<td>0.845</td>
<td>0.929</td>
<td>0.710</td>
<td>0.949</td>
</tr>
<tr>
<td>8</td>
<td>0.814</td>
<td>0.827</td>
<td>0.841</td>
<td>0.828</td>
<td>0.802</td>
</tr>
<tr>
<td>( \Pi )</td>
<td>1.97E-07</td>
<td>1.89E-07</td>
<td>1.89E-07</td>
<td>2.00E-07</td>
<td>1.85E-07</td>
</tr>
</tbody>
</table>

IPS based estimation typically yields values 0.0 for \( k \geq 4 \)
Reach probabilities estimated through 10 runs

<table>
<thead>
<tr>
<th>Event</th>
<th>Mean</th>
<th>Std. dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>MTC</td>
<td>1.0</td>
<td>4.8E-03</td>
</tr>
<tr>
<td>STC</td>
<td>2.35E-04</td>
<td>5.0E-04</td>
</tr>
<tr>
<td>MSI</td>
<td>2.57E-06</td>
<td>3.9E-07</td>
</tr>
<tr>
<td>NMAC</td>
<td>2.82E-07</td>
<td>4.5E-08</td>
</tr>
<tr>
<td>MAC</td>
<td>1.91E-07</td>
<td>1.6E-08</td>
</tr>
</tbody>
</table>

Contribution to reach probability

<table>
<thead>
<tr>
<th>Global comm.</th>
<th>DM-loop</th>
<th>Share %</th>
</tr>
</thead>
<tbody>
<tr>
<td>up</td>
<td>up</td>
<td>0.5</td>
</tr>
<tr>
<td>up</td>
<td>down</td>
<td>1.9</td>
</tr>
<tr>
<td>down</td>
<td>up</td>
<td>97.6</td>
</tr>
<tr>
<td>down</td>
<td>down</td>
<td>0.002</td>
</tr>
</tbody>
</table>
Two-aircraft encounter and dependable technical systems
Eight aircraft encounter using IPS

<table>
<thead>
<tr>
<th>Level</th>
<th>1&lt;sup&gt;st&lt;/sup&gt; IPS</th>
<th>2&lt;sup&gt;nd&lt;/sup&gt; IPS</th>
<th>3&lt;sup&gt;rd&lt;/sup&gt; IPS</th>
<th>4&lt;sup&gt;th&lt;/sup&gt; IPS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>2</td>
<td>0.528</td>
<td>0.529</td>
<td>0.539</td>
<td>0.533</td>
</tr>
<tr>
<td>3</td>
<td>0.426</td>
<td>0.429</td>
<td>0.424</td>
<td>0.431</td>
</tr>
<tr>
<td>4</td>
<td>0.033</td>
<td>0.036</td>
<td>0.035</td>
<td>0.037</td>
</tr>
<tr>
<td>5</td>
<td>0.175</td>
<td>0.180</td>
<td>0.183</td>
<td>0.181</td>
</tr>
<tr>
<td>6</td>
<td>0.267</td>
<td>0.158</td>
<td>0.177</td>
<td>0.144</td>
</tr>
<tr>
<td>7</td>
<td>0.150</td>
<td>0.268</td>
<td>0.281</td>
<td>0.427</td>
</tr>
<tr>
<td>8</td>
<td>0.000</td>
<td>0.009</td>
<td>0.233</td>
<td>0.043</td>
</tr>
<tr>
<td>Product of fractions</td>
<td>0.0</td>
<td>5.58 \cdot 10^{-7}</td>
<td>1.67 \cdot 10^{-5}</td>
<td>4.01 \cdot 10^{-6}</td>
</tr>
</tbody>
</table>
Two-aircraft vs. eight-aircraft encounter

Event probability for aircraft # 1

Safety related events

MTC  STC  MSI  NMAC  MAC
Eight-aircraft encounter: Baseline PF response vs. Fast PF response
Random traffic scenario, high density

- Eight aircraft per packed container
  - 3 times as dense above Frankfurt on 23rd July ‘99
  - IPS, 10,000 particles, 30 hours per IPS run
IPS runs for random traffic scenario

<table>
<thead>
<tr>
<th>Level</th>
<th>1&lt;sup&gt;st&lt;/sup&gt; IPS</th>
<th>2&lt;sup&gt;nd&lt;/sup&gt; IPS</th>
<th>3&lt;sup&gt;rd&lt;/sup&gt; IPS</th>
<th>4&lt;sup&gt;th&lt;/sup&gt; IPS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.922</td>
<td>0.917</td>
<td>0.929</td>
<td>0.926</td>
</tr>
<tr>
<td>2</td>
<td>0.567</td>
<td>0.551</td>
<td>0.560</td>
<td>0.559</td>
</tr>
<tr>
<td>3</td>
<td>0.665</td>
<td>0.666</td>
<td>0.674</td>
<td>0.676</td>
</tr>
<tr>
<td>4</td>
<td>0.319</td>
<td>0.331</td>
<td>0.323</td>
<td>0.321</td>
</tr>
<tr>
<td>5</td>
<td>0.370</td>
<td>0.367</td>
<td>0.371</td>
<td>0.379</td>
</tr>
<tr>
<td>6</td>
<td>0.181</td>
<td>0.158</td>
<td>0.162</td>
<td>0.171</td>
</tr>
<tr>
<td>7</td>
<td>0.130</td>
<td>0.209</td>
<td>0.174</td>
<td>0.145</td>
</tr>
<tr>
<td>8</td>
<td>0.067</td>
<td>0.005</td>
<td>0.094</td>
<td>0.066</td>
</tr>
<tr>
<td>Product of fractions</td>
<td>$6.42 \cdot 10^{-5}$</td>
<td>$6.76 \cdot 10^{-6}$</td>
<td>$1.11 \cdot 10^{-4}$</td>
<td>$6.99 \cdot 10^{-5}$</td>
</tr>
</tbody>
</table>
Random high traffic: Uncontrolled vs. AMFF controlled

Event probability per flight hour for aircraft # 1

Safety related events
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Conclusions

- Thanks to IPS developments it has been shown that uncoordinated conflict resolution falls short in safely accommodating high en route traffic demand.

- Follow-up work on risk assessment:
  - Continue developments of IPS and HHIPS
    - Extending Convergence Proof
    - Monte Carlo Markov Chain
    - Traffic complexity prediction
  - Evaluate advanced airborne self separation concept
  - Include ACAS in simulation model
  - Validation of assessed risk level
Validation of assessed risk level

- Simulation model ≠ Reality
- Identify the differences
- Assess each difference individually (and conditionally)
  - use of statistical data and expert knowledge
- Assess model parameter sensitivities by Monte Carlo simulations
- Evaluate effect of each assumption at simulated risk level
  - use of statistical data and expert knowledge
- Evaluate combined effects of all model assumptions
  - Typical output: expected risk and 95% area
- Improve simulation model for large differences
To be continued

http://iFly.nlr.nl