OBSERVER DESIGN FOR DISCRETE-TIME LINEAR SWITCHING SYSTEMS [1]

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Abstract: Observability was previously introduced and characterized for the class of linear switching systems. In the discrete-time case, detection requires a finite, non-zero, number of time instants, while in continuous-time detection can be achieved in an arbitrarily small amount of time. To address this issue, $\Delta-$observability was introduced, which is based on the reconstruction of the hybrid state at least in some - but not necessarily all - time intervals. In this paper, we build upon these results to give a definition of a hybrid observer for discrete-time switching systems and to propose an observer design technique.

Keywords: linear switching systems, observability, hybrid state reconstruction, hybrid observer

1. INTRODUCTION

Different notions of observability of hybrid systems have been given in the literature, depending on the class of systems under consideration and on the knowledge that is assumed at the output (see e.g. (Vidal *et al.*, 2003), (De Santis *et al.*, 2003), (Babaali and Egerstedt, 2004), (De Santis *et al.*, 2006), (Bemporad *et al.*, 2000)). The design of hybrid observers was investigated in (Balluchi *et al.*, 2002), where a methodology for dynamical observers of hybrid systems with linear continuous-time dynamics was proposed. In this approach, the complete state (discrete location and continuous state) is reconstructed from the knowledge of the inputs and outputs of a hybrid plant. The synthesis of hybrid observers was addressed in a set of cases characterized by the amount of information available on the current location ranging from the case of complete knowledge (i.e. the hybrid plant produces as discrete output its current location), treated for instance in (Alessandri and Coletta, 2001), to the case of absence of discrete output information (i.e. the hybrid plant produces no discrete output), considered e.g. in (Bemporad *et al.*, 2000; Ferrari-Trecate *et al.*, 2000). A definition of a hybrid observer for the class of autonomous piecewise affine systems was given in (Collins and van Schuppen, 2004).

In (De Santis *et al.*, 2007) the definitions and results of (De Santis *et al.*, 2003) were adapted to discrete-time linear switching systems and conditions were given to test observability. In the discrete-time case, detection requires a finite, non-zero, number of time instants, while in continuous-time detection can be achieved in an arbitrarily small amount of time. To address this issue, $\Delta-$observability was introduced, which is based

[1] This work was partially supported by the HYCON Network of Excellence, contract number FP6-IST-511368, European Commission under STREP project n. TREN/07/FP6AE/S07.71574/037180 IFLY and by Ministero dell'Istruzione, dell'Universita' e della Ricerca under Projects MACSI and SCEF (PRIN05)

on the reconstruction of the hybrid state at least in some - but not necessarily all - time intervals. In this paper, we build upon the results of (De Santis *et al.*, 2007) to give a definition of a hybrid observer for discrete-time switching systems and to propose an observer design technique.

The paper is organized as follows. In Section 2, we introduce discrete-time linear switching systems and the notions of $\Delta-$observability. In Section 3, an observer design technique is presented. Finally, Section 4 offers some concluding remarks.

Notation. We denote by $\mathbb{I}$ the set of integers and by $\mathbb{R}$ the set of reals. For any $a, b \in \mathbb{I}$ for which $a \leq b$ we set $[a, b] = \{z \in \mathbb{I} : a \leq z \leq b\}$. The symbol $\mathrm{Im}\,(M)$ denotes the range space of some matrix M. For a function $f : \mathbb{I} \to \mathbb{R}^n$, the symbol $\mathbf{f}|_{[a,b]}$ with $a \leq b$ denotes the vector

$$\begin{pmatrix} f(a) \\ f(a+1) \\ \vdots \\ f(b) \end{pmatrix} \in \mathbb{R}^{n(b-a+1)}$$

2. PRELIMINARY DEFINITIONS AND RESULTS

In this section, we recall some definitions and results for the class of *discrete–time linear switching systems* (De Santis *et al.*, 2007). The inputs of a linear switching system are a discrete and unknown disturbance σ and a continuous control input u. The hybrid state ξ is composed of two components: the discrete state i belonging to a finite set Q and the continuous state x belonging to the linear space $\mathbb{R}^{n_i}$, whose dimension n_i depends on i. The hybrid output has a discrete and a continuous component as well, the former associated to the discrete states and the latter associated to the continuous state. The evolution of the discrete state is governed by a Finite State Machine; a transition $e = (i, \sigma, h)$ may occur at time t from the discrete state i to the discrete state h, if the discrete disturbance σ occurs at time t. The evolution of the continuous state is described by a set of discrete–time linear dynamical systems, controlled by the continuous input u, and whose matrices depend on the current discrete state i. Whenever a transition e occurs, the continuous state x is instantly reset to a new value $R(e)x$, where $R(e)$ is a matrix depending on the transition e.

For simplicity, in this paper we assume that no discrete signal is available at the output of the switching system, and that a transition is defined for each pair i, h. More formally,

Definition 1. A discrete–time linear switching system $\mathcal{S}$ is a tuple

$$\left(\Xi, \Theta, \mathbb{R}^l, S, E, R\right), \tag{1}$$

where:

- $\Xi = \bigcup_{i_i \in Q} \{i\} \times \mathbb{R}^{n_i}$ is the hybrid state space, where:
  - ◦ $Q = \{1, 2, \ldots, N\}$ is the discrete state space;
  - ◦ $\mathbb{R}^{n_i}$ is the continuous state space associated with the discrete state $i \in Q$;
- $\Theta = \Sigma \times \mathbb{R}^m$ is the hybrid input space, where:
  - ◦ $\Sigma = \{\sigma_j, j \in J\}$ is the discrete disturbance space, $J = \{1, 2, \ldots, N_1\}$;
  - ◦ $\mathbb{R}^m$ is the continuous control input space;
- $\mathbb{R}^l$ is the continuous output space;
- S is a map associating to each discrete state $i \in Q$ the linear dynamical control system:
$$S(i) : \begin{cases} x(t+1) = A_i x(t) + B_i u(t), \\ y(t) = C_i x(t), \end{cases}$$
  where $x(t) \in \mathbb{R}^{n_i}$ is the continuous state, $u(t) \in \mathbb{R}^m$ is the continuous control input, $y(t) \in \mathbb{R}^l$ is the continuous output, $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times m}$ and $C_i \in \mathbb{R}^{l \times n_i}$;
- $E \subset Q \times \Sigma \times Q$ is a collection of transitions;
- R is the reset function associating to every $e = (i, \sigma, h) \in E$ the reset matrix $R(e) \in \mathbb{R}^{n_h \times n_i}$.

We now define the semantics of linear switching systems. We assume that *the discrete disturbance is not available for measurement.* Following (Lygeros *et al.*, 1999), a *hybrid time basis* τ is an infinite or finite sequence of sets $I_j = \{t \in \mathbb{I} : t_j \leq t \leq t'_j\}$, with $t'_j > t_j$ and $t'_j = t_{j+1}$; let be $card(\tau) = L+1$. If $L < \infty$, then t'_L can be finite or infinite. A hybrid time basis τ is said to be *finite*, if $L < \infty$ and $t'_L < \infty$ and *infinite*, otherwise. Given a hybrid time basis τ, time instants t'_j are called *switching times*. Denote by $\mathcal{T}$ the set of all hybrid time bases. The switching system temporal evolution is defined as follows.

Definition 2. (Linear switching system execution) An execution χ of a linear switching system $\mathcal{S}$ is a collection:

$$(\xi_0, \tau, \sigma, u, \xi, \eta),$$

with hybrid initial state $\xi_0 \in \Xi$, hybrid time basis $\tau \in \mathcal{T}$, discrete disturbance $\sigma : \mathbb{I} \to \Sigma$, control input $u : \mathbb{I} \to \mathbb{R}^m$, hybrid state evolution $\xi : \mathbb{I} \times \mathbb{I} \to \Xi$ and output evolution $\eta : \mathbb{I} \to \mathbb{I} \to \mathbb{R}^l$. The hybrid state evolution ξ is defined as follows:

$$\begin{aligned} &\xi\,(t_0, 0) = \xi_0, \\ &\xi\,(t, j) = (q\,(j)\,, x(t, j))\,,\ t \in I_j, j = 0, ..., L, \\ &\xi\,(t_{j+1}, j+1) = (q\,(j+1)\,, R(e_j)x(t'_j, j)), \\ &\qquad\qquad\qquad\qquad\qquad j = 0, ..., L-1, \end{aligned}$$

where $q : \mathbb{I} \to Q$ and for any $j = 0, 1, ..., L$, $e_j = (q\,(j)\,, \sigma\,(j)\,, q\,(j+1)) \in E$ and $x\,(t, j)$ is the (unique) solution at time $t \in I_j$ of the dynamical system $S(q(j))$, with initial time t_j, initial condition $x(t_j, j)$ and control law u. The output evolution of $\mathcal{S}$ is specified by the function $\eta : \mathbb{I} \to \mathbb{R}^l$, which for any $j = 0, 1, ..., L$ is defined as:

$$\eta(t) = C_i x(t_j, j), t \in [t_j, t'_j - 1], \qquad (2)$$

where C_i is the output matrix associated with the current discrete state $q(j) = i$.

In linear system theory, observability deals with the reconstruction of the state, on the basis of the knowledge of the continuous input and of the continuous output that is accessible from the environment. In the following, we generalize those notions to the class of linear switching systems.

According to the definition of the function η, the transition from one discrete state to another may not be visible from the output.

The definition of observability we propose is based on the existence of an input–output experiment such that the hybrid state is reconstructed, at least in some time intervals of the time basis.

Definition 3. Given a nonnegative integer Δ, a linear switching system $\mathcal{S}$ is $\Delta-$*observable* if there exist a control input $\widehat{u} : \mathbb{I} \to \mathbb{R}^m$ and a function $\widehat{\xi} : \mathbb{R}^{l(\Delta+1)} \times \mathbb{R}^{m\Delta} \to 2^{\Xi}$ such that

$$\widehat{\xi}\left(\eta|_{[t-\Delta,t]}, \widehat{u}|_{[t-\Delta,t-1]}\right) = \{\xi(t,j)\}$$

$\forall t \in \left[t_j + \Delta, t'_j - 1\right]$, for any execution χ with control input $\widehat{u}$, for any $j \in \{0, 1, ..., L\}$ such that $t'_j - t_j \geq \Delta + 1$.

Moreover, $\mathcal{S}$ is said to be *observable* if there exists a nonnegative integer Δ such that $\mathcal{S}$ is $\Delta-$observable.

The available information at some $t \in \left[t_j, t'_j - 1\right]$, given by the input $\widehat{u}_{[t-\Delta,t-1]}$ and the observations $\eta_{[t-\Delta,t]}$, could be compatible with more than one current hybrid state. The above definition requires that the current hybrid state is unambiguously determined at each time instant in the interval $\left[t_j + \Delta, t'_j - 1\right]$.

Since in general a system cannot be $\Delta-$observable with $\Delta = 0$, because in that case only the value $\eta(t)$ would be available at the output, we assume $\Delta \geq 1$ without loss of generality.

By specializing Definition 3 to the reconstruction of the discrete component of the hybrid state *only*, the following definition is obtained.

Definition 4. Given an integer $\Delta \geq 1$, a linear switching system $\mathcal{S}$ is $\Delta-$*location observable for an input function* $\widehat{u} : \mathbb{I} \to \mathbb{R}^m$, if there exists a function $\widehat{q} : \mathbb{R}^{l(\Delta+1)} \times \mathbb{R}^{m\Delta} \to 2^Q$ such that

$$\widehat{q}\left(\eta|_{[t-\Delta,t]}, \widehat{u}|_{[t-\Delta,t-1]}\right) = \{q(j)\}$$

$\forall t \in \left[t_j + \Delta, t'_j - 1\right]$, for any execution χ with control input $\widehat{u}$, for any $j \in \{0, 1, ..., L\}$ such that $t'_j - t_j \geq \Delta + 1$.

The system $\mathcal{S}$ is called $\Delta-$*location observable* if there exists an input function $\widehat{u}$ for which it is location observable. The system $\mathcal{S}$ is called *location observable* if there exists Δ for which it is $\Delta-$location observable.

For a given Δ, $\mathcal{U}^{\Delta}$ denotes the set of input functions for which $\mathcal{S}$ is $\Delta-$location observable.

A well–known assumption on the behavior of switching systems is the existence of a dwell time (Morse, 1996). Formally, a non negative $\delta \in \mathbb{I}$ is said to be a dwell time for a switching system $\mathcal{S}$ if any execution $\chi = (\xi_0, \tau, \sigma, u, \xi, \eta)$ generated by $\mathcal{S}$ is characterized by the following property:

$$t'_j - t_j \geq \delta,$$

for any $[t_j, t'_j] \in \tau$. If a $\Delta-$observable switching system $\mathcal{S}$ has a dwell time $\delta \geq \Delta + 1$, then each execution is such that $t'_j - t_j \geq \Delta + 1$, for any $j \in \{0, 1, ..., L\}$. Hence, in that case, the conditions of Definition 4 hold for *any* execution with control input $\widehat{u} \in \mathcal{U}^{\Delta}$, for any $j \in \{0, 1, ..., L\}$.

We can state the following:

Theorem 1. (De Santis *et al.*, 2007)Given a linear switching system $\mathcal{S}$

- **i)** $\mathcal{S}$ is location observable if and only if $\forall (i, h) \in J \times J, \exists k \in \mathbb{I}, 0 \leq k < n_i + n_h : C_i A_i^k B_i \neq C_h A_h^k B_h$.
- **ii)** $\mathcal{S}$ is observable if and only if it is location observable and (A_i, C_i) is observable $\forall i \in J$.
- **iii)** if $\mathcal{S}$ is $\Delta-$location observable, then $\mathcal{S}$ is $\widehat{\Delta}-$observable for any $\widehat{\Delta} \geq \Delta + \max_{i \in J} n_i$.

If a system is observable, an interesting question is how to construct a function $\widehat{q}$ that reconstructs the current discrete state at least in the time interval $\left[t_j + \Delta, t'_j - 1\right]$. To do this, we need to use the available information on the input and observable output to determine all the discrete states that are compatible with it. However, over some time interval of length Δ, the dynamics of a node i may be indistinguishable from the dynamics resulting from a switching that occurred in that interval between node i and another node h. Then, since the switching times t_j are not known a priori, it would not be possible to understand whether the discrete state is i or h.

This motivates the following definition:

Definition 5. Given a linear switching system $\mathcal{S}$, a function $\widehat{\xi} : \mathbb{R}^{l(\Delta+1)} \times \mathbb{R}^{m\Delta} \to 2^{\Xi}$ is called

$\Delta-$*hybrid state observer of* $\mathcal{S}$, for an input function $\widehat{u}$, if for any execution χ with control input $\widehat{u}$, for any $j \in \{0,1,...,L\}$, such that $t'_j - t_j \geq \Delta+1$, the following three conditions hold:

i) $\widehat{\xi}\left(\eta|_{[t-\Delta,t]}, \widehat{u}|_{[t-\Delta,t-1]}\right) = \{\xi(t,j)\}$, $\forall t \in [t_j+\Delta, t'_j-1]$;

ii) there exists a time $\widehat{t}_j \in [t_j, t_j+\Delta]$ at which the occurrence of a switching is detected;

iii) if $\widehat{\xi}\left(\eta|_{[t-\Delta,t]}, \widehat{u}|_{[t-\Delta,t-1]}\right) = \{(q,x)\}$, for some $t \in \left[\widehat{t}_j, t'_j-1\right]$, then $\xi(t,j) = (q,x)$.

A function $\widehat{q} : \mathbb{R}^{l(\Delta+1)} \times \mathbb{R}^{m\Delta} \rightarrow 2^Q$ is called $\Delta-$*discrete state observer of* $\mathcal{S}$, for an input function $\widehat{u}$, if for any execution χ with control input $\widehat{u}$, for any $j \in \{0,1,...,L\}$, such that $t'_j - t_j \geq \Delta+1$, the following three conditions hold:

i') $\widehat{q}\left(\eta|_{[t-\Delta,t]}, \widehat{u}|_{[t-\Delta,t-1]}\right) = \{q(j)\}$, $\forall t \in [t_j+\Delta, t'_j-1]$;

ii') there exists a time $\widehat{t}_j \in [t_j, t_j+\Delta]$ at which the occurrence of a switching is detected;

iii') if $\widehat{q}\left(\eta|_{[t-\Delta,t]}, \widehat{u}|_{[t-\Delta,t-1]}\right) = \{i\}$, for some $t \in \left[\widehat{t}_j, t'_j-1\right]$, then $q(j) = i$.

Roughly speaking, a switching may not be instantaneously identified. However, it has to be detected at some time in the interval $[t_j, t_j+\Delta]$. Hence, the discrete observer may return an erroneous estimation of the current state if the current data are compatible with the discrete state before the switching. After detection of a switching, if the function $\widehat{q}$ returns a singleton, then this singleton has to be equal to the current discrete state.

For the existence of a $\Delta-$hybrid state observer, the system $\mathcal{S}$ has to be $\Delta-$observable. Conversely, $\Delta-$observability does not imply the existence of a $\Delta-$hybrid state observer. As an example, consider a system $\mathcal{S}$ with one discrete mode q, with $E = \{e = (q,\sigma,q)\}$, and observable dynamic system $S(q)$ described by

$$x(t+1) = Ax(t)$$
$$y(t) = x(t)$$

System $\mathcal{S}$ is obviously $\Delta-$observable, for any $\Delta \geq 0$. However, condition ii) may not hold for any function $\widehat{\xi}$ satisfying the observability condition: in fact, even if $R(e)$ is not the identity, it may happen that $x(t'_j,j) = R(e)x(t'_j,j) = x(t_{j+1},j+1)$ and the available information does not allow the reconstruction of the switching that occurred at time t_j. As another example, consider a $\Delta-$observable switching $\mathcal{S}$ with no in-loop transitions i.e. $E^{\circlearrowleft} = \{e = (q_i,\sigma,q_h) \in E : q_i = q_h\ \} = \varnothing$. Then, for a sufficiently large Δ, conditions i) and ii) (resp. i') and ii')) are satisfied while condition iii) (resp. iii')) may not hold if $\widehat{t}_j \in [t_j, t_j+\Delta-1]$.

3. OBSERVER DESIGN

As done in (Balluchi *et al.*, 2002), we decompose the design of a hybrid observer of $\mathcal{S}$ into two steps: we first design the discrete state observer (i.e. the function $\widehat{q}$ introduced in Definition 5) and then the hybrid state observer. In the next subsection, we will describe each component.

3.1 The discrete state observer

In what follows, we always assume that $\mathcal{S}$ is Δ_s-location observable and that the input function is in $\mathcal{U}^{\Delta_s}$. We also assume that there are no in–loop transitions. Then, for any input function in $\mathcal{U}^{\Delta_s}$, if a switching occurred at some time t' from a known state q_i to an unknown state q_h, at some time $t \in [t', t'+\Delta_s]$ the observations $\eta|_{[t-\Delta_s,t]}$ are guaranteed to indicate that a switching occurred and that the switching time was in the interval $[t-\Delta_s, t]$.

Δ_s-location observability implies Δ- location observability, for all $\Delta \geq \Delta_s$. An estimation of a lower bound for Δ_s was given in (De Santis *et al.*, 2007). This estimation does not play any particular role here, it is therefore omitted.

For a given positive integer d, define the matrices $\mathbf{F}^{(d,i)} = diag\{C_i\}\mathbf{M}^{(d,i)} \in \mathbb{R}^{(d+1)l\times n_i}$ and $\mathbf{G}^{(d,i)} = diag\{C_i\}\mathbf{N}^{(d,i)} \in \mathbb{R}^{dl\times dn_i}$, where

$$\mathbf{M}^{(d,i)} = \begin{pmatrix} I \\ A_i \\ \vdots \\ A_i^d \end{pmatrix}$$
$$\mathbf{N}^{(d,i)} = \begin{pmatrix} 0 & \dots & \dots & \dots & \dots \\ B_i & 0 & \dots & \dots & \dots \\ A_iB_i & B_i & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ A_i^{d-1}B_i & A_i^{d-2}B_i & \dots & A_iB_i & B_i \end{pmatrix}$$

and the set $\mathcal{F}_t^d$

$$\left\{i \in Q : \eta|_{[t-d,t]} - \mathbf{G}^{(d,i)}\, u|_{[t-d,t-1]} \in \mathrm{Im}\left(\mathbf{F}^{(d,i)}\right)\right\} \tag{3}$$

which is the set of all discrete states compatible with the observations $\eta|_{[t-d,t]}$ and the inputs $u|_{[t-d,t-1]}$, *under the hypothesis that no switchings occurred in the time interval* $[t-d,t]$ from state i to another discrete state. If this hypothesis does not hold, then the set $\mathcal{F}_t^d$ can be empty.

Set

$$\mathcal{F}_t^0 = \{i \in Q : \eta(t) \in \mathrm{Im}\,(C_i)\}$$

Define the functions $\phi : \mathbb{I} \to 2^Q$ and $\mathbf{f} : \mathbb{I} \to \{0,1\}$ as follows:

$t_s = 0, \mathbf{f}(0) = 1$
while $t_s < \infty$,
  $\phi(t_s) = \mathcal{F}^0_{t_s},\ t = t_s + 1,\ \Delta_t = 1$
  while $\mathcal{F}^{\Delta_t}_t \neq \varnothing$ and $\mathcal{F}^{\Delta_t}_t \subset \mathcal{F}^{\Delta_t - 1}_{t-1}$
    $\phi(t) = \mathcal{F}^{\Delta_t}_t$
    $\mathbf{f}(t) = 0$
    $t = t + 1$
    $\Delta_t = \min\{t - t_s, \Delta_s\}$
  endwhile
  $\mathbf{f}(t) = 1$
  $t^+_s = t$
  $t_s = t^+_s$
endwhile.

Define

$$\widehat{q}\left(\eta|_{[t-\Delta,t]}, \widehat{u}|_{[t-\Delta,t-1]}\right) = \phi(t) \tag{4}$$

The next result states that $\widehat{q}$ satisfies the conditions of Definition 5, and the time instants t at which $\mathbf{f}(t) = 1$ are the times at which the switchings are detected.

Proposition 1. Let the system $\mathcal{S}$ be Δ_s-location observable and assume there is a dwell time $\delta \geq \Delta + 1$, $\Delta = 2\Delta_s$. The function $\widehat{q}$ defined in 4 is a $\Delta-$discrete state observer of $\mathcal{S}$, for any $u \in \mathcal{U}^{\Delta_s}$.

Proof. Since $\mathcal{S}$ is Δ_s-location observable, at each $t \in \left[t_s + \Delta_s, t'_j - 1\right]$ the observations $\eta|_{[t-\Delta_s,t]}$ and the inputs $u|_{[t-\Delta_s,t-1]}$ are compatible only with the discrete state $q(j)$. In $\left[t_s,\ t'_j - 1\right]$ no switchings occurred, and since by definition $\phi(t)$ is the set of all discrete states compatible with the observations $\eta|_{[t-\Delta_s,t]}$ and the inputs $u|_{[t-\Delta_s,t-1]}$, $\phi(t) = \{q(j)\}$, $\forall t \in \left[t_s + \Delta_s, t'_j - 1\right]$. Hence $\phi(t) = \{q(j)\}$, $\forall t \in \left[t_j + 2\Delta_s, t'_j - 1\right]$. We now prove that given t_s, the updated value t^+_s in the definition of ϕ is the time instant at which from the output it is possible to deduce that a switching occurred in the time interval $\left[t^+_s - \Delta_{t^+_s}, t^+_s + 1\right]$, $\Delta_{t^+_s} = \min\{t^+_s - t_s, \Delta_s\}$. Suppose that $t_s \in I_j$ is the time instant at which it is possible to deduce from the output that a switching occurred in the interval $[t_s - \Delta_{t_s}, t_s]$. Then, in the interval $[t_s, t_s + \Delta_s]$ no switching can occur, and at any $t \in [t_s, t_s + \Delta_s]$, $\mathcal{F}^{\Delta_t}_t \subset \mathcal{F}^{\Delta_t - 1}_{t-1}$ and $\mathcal{F}^{\Delta_t}_t \neq \varnothing$. If ϕ returns a singleton at some time in the interval $[t_s, t_s + \Delta_s]$, this singleton is the current discrete state. Moreover, at any $t \in \left[t_s + \Delta_s, t'_j - 1\right]$ the function ϕ returns the correct value for the state, and it returns the same value for $t \geq t'_j$, until such value is no more compatible with the available information, in which case $\mathcal{F}^{\Delta_t}_t = \varnothing$ or $\mathcal{F}^{\Delta_t}_t \subsetneq \mathcal{F}^{\Delta_t - 1}_{t-1}$. At this point t^+_s is equal to t, and we are sure that a switching occurred in the time interval $\left[t^+_s - \Delta_{t^+_s}, t^+_s\right]$. Since t_s is initially set at 0 and we can suppose that 0 is a switching time, then by induction the result follows, i.e. t^+_s is the time instant at which it is possible to deduce from the output that a switching occurred in the interval $\left[t^+_s - \Delta_{t^+_s}, t^+_s 1\right]$ and $\widehat{q}$ satisfies the conditions of Definition 5. ■

The estimation of Δ for which it is possible to define a discrete observer can be refined, with respect to the estimation we have considered here, i.e. $\Delta \geq 2\Delta_s$. Moreover, a different and more sophisticated design for the discrete observer is possible, for example to increase the speed of convergence to the actual discrete state. The question is then to find the best compromise between the performance and the simplicity of the observer.

3.2 The hybrid state observer

Let $\widehat{q}$ be a $\Delta'-$discrete state observer for $\mathcal{S}$, with a given Δ', and assume a dwell time $\delta \geq \Delta + 1$, $\Delta = \max\{\Delta', \max_{i\in Q} n_i\}$. The discrete state observer at time t gives the values of $\phi(t)$ and $\mathbf{f}(t)$, as described in the previous subsection. We assume that (A_i, C_i) is observable, $\forall i \in Q$.

The design of the hybrid state observer leverages the information returned by the discrete state observer, as described below.

Consider the function $\xi^O : \mathbb{I} \to 2^\Xi$, defined as:

$$\xi^O(t) = \bigcup_{i\in\phi(t)} \{i\} \times X^i(t)$$

where, if $\mathbf{f}(t) = 1$, then

$$X^i(t) = \{x \in \mathbb{R}^{n_i} : C_i x = \eta(t)\}$$

and if $\mathbf{f}(t) = 0$ then $X^i(t)$ is the set of continuous states compatible with the dynamic system S_i, the input $u|_{\left[t-\overline{\Delta},t-1\right]}$ and the observations $\eta|_{\left[t-\overline{\Delta},t\right]}$, $\overline{\Delta} = \min\{t - t_s, \max_{i\in Q} n_i\}$, where $t_s < t$ is the last time instant d at which $\mathbf{f}(d)$ is equal to 1. Formally,

$$X^i(t) = \mathbf{M}^{(\overline{\Delta},i)} X^i_0(t) + \mathbf{N}^{(\overline{\Delta},i)}\ u|_{\left[t-\overline{\Delta},t-1\right]}$$
$$X^i_0(t) = \left\{x : \mathbf{F}^{(\overline{\Delta},i)} x = \eta|_{\left[t-\overline{\Delta},t\right]} - \mathbf{G}^{(\overline{\Delta},i)}\ u|_{\left[t-\overline{\Delta},t-1\right]}\right\}$$

Given $\phi(t)$ and $\mathbf{f}(t)$, set

$$\widehat{\xi}\left(\eta|_{[t-\Delta,t]}, \widehat{u}|_{[t-\Delta,t-1]}\right) = \xi^O(t) \tag{5}$$

We now show that $\widehat{\xi}$, defined in 4, satisfies the requirements of Definition 5, for a suitable value of Δ.

Proposition 2. Let the system $\mathcal{S}$ be Δ_s-location observable and assume there is a dwell time $\delta \geq \Delta + 1$, $\Delta = 2\Delta_s + \max_{i\in Q} n_i$. The function $\widehat{\xi}$, defined in 5 is a $\Delta-$discrete state observer of $\mathcal{S}$, for any $u \in \mathcal{U}^{\Delta_s}$.

Proof. Consider any execution with input function $u \in \mathcal{U}^{\Delta_s}$, and a time interval $\left[t_j, t_j'\right]$. By assumption, $t_j' - t_j \geq \Delta$, for all j. Let t_s be a time in $\left[t_j, t_j'\right]$ such that $\mathbf{f}(t_s) = 1$. Since $\mathcal{S}$ is Δ_s-location observable, then t_s exists, is unique, and belongs to the interval $[t_j, t_j + \Delta_s]$. At time $t_j + 2\Delta_s$, the current discrete state has been reconstructed, at time $t_s + \max_{i \in Q} n_i$, which by assumption belongs to $\left[t_s, t_j' - 1\right]$, the value for the current continuous state has been reconstructed, and, if the function ξ^O gives a singleton at some time in $\left[t_s, t_j' - 1\right]$, such a singleton coincides with the current hybrid state, since the dynamics are observable, and no switchings occurred in the interval $\left[t_s, t_j' - 1\right]$. ■

The hybrid observer we proposed above simply computes at each step all the hybrid states compatible with the information obtained from the discrete observer and with the observation window. When a switching is detected, i.e. when $\mathbf{f}(t) = 1$, the size of the observation window is reset to zero, to be sure that the measurements that are input to the observer are produced by the same dynamical system, so that they are not subject to uncertainties due to a non exact reconstruction of the switching time.

The proposed technique can be improved. For example, we can compute at time t_s an estimation $\bigcup_{i \in \phi(t_s)} \{q_i\} \times \widehat{X}^i(t_s)$ of the set of states compatible with the past observations, rather than assuming that at t_s the set of possible states is bounded by $\bigcup_{i \in \phi(t_s)} \{q_i\} \times X^i(t_s)$, $X^i(t_s) = \{x \in \mathbb{R}^{n_i} : C_i x = \eta(t_s)\}$. This refinement implies a better estimation of the transient continuous state evolution and possibly a faster convergence of the hybrid observer.

4. CONCLUSIONS

In this paper, we proposed a definition of a hybrid observer for discrete-time switching systems and we presented an algorithm for observer design. Future work includes the development of "customized" techniques according to problem-specific requirements of speed of convergence, precision, or simplicity. Possible extensions can be considered by removing the assumption of observability on the dynamical systems, and addressing detectability of the switching system.

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